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A DIGITAL METHOD OF LOCATING THE POLES AND ZEROS OF AN IMPEDANCE FUNCTION

A THESIS

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A DIGITAL METHOD OF LOCATING THE POLES AND ZEROS OF AN IMPEDANCE FUNCTION

Approved: ____________________

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FOREWORD

The aim of this investigation was to give attention not only to the theoretical, but also to the practical aspects of the problem. Therefore, a great deal of time was spent running test problems on a digital computer which might otherwise have been spent on the theoretical analysis. As a result of this approach, the method developed is closer to being a useful tool. The investigation into this subject has not been exhaustive, and further study would certainly be rewarding.

The staff of the Rich Electronic Computer Center, especially Alton P. Jensen, were most cooperative. When difficulty was encountered with a program, their assistance was always available.

I am grateful to my advisor, Dr. Kendall Su, for his criticism of the original draft of this thesis, and for his suggestions on the organization of the material. Both were indispensable.

My wife, Emily, made this study possible. Not only did she type it, but she also endured long evenings while I worked on it. It is to her that it is dedicated.

TABLE OF CONTENTS

LIST OF SYMBOLS USED

Symbol	Definition
c_n	Coefficient of $(s-s_o)^n$ in a power series.
h	Spacing between data points.
k_ν	Residue of pole at s_ν.
n	Order of derivative.
p	Multiplicity of a pole.
r	Radius of convergence of a series.
s	Complex variable $\sigma+j\omega$.
s_ν	Singularity nearest to given point in s plane.
$s_\nu*$	Estimated location of singularity at s_ν.
$s_{\nu\pm1}$	Singularities nearest to s_ν.
$Z(s)$	A function of s.
$Z^{(n)}(s)$	n-th derivative of Z(s) with respect to s.

LIST OF ILLUSTRATIONS

SUMMARY

Many functions encountered in engineering work fall into the class termed rational functions of a complex variable. Often, the value of such a function is known along a line segment in the s plane (usually a portion of the positive imaginary axis) and the analytical expression for the function is desired. The purpose of this study was to develop a method of finding this expression.

If the value of a function along a finite line segment is known, a power series may be written at a point on this line which will be identical with the function within the circle of convergence of the series. The series will converge within a circle with radius extending to the nearest singularity of the function. Three formulas for finding this radius are presented. These formulas may be interpreted as giving the distance from a point in the s plane to the nearest singularity. One of these formulas is chosen for further study, and it is shown that for the case of rational functions, the direction as well as the distance to the nearest pole is available.

The practical use of this formula involves errors from three sources; from the inexact methods by which the derivatives must be obtained, from the fact that one may not take an infinite number of derivatives, and from computational

errors. Where possible, the magnitude of error is estimated.

Several test problems were run on a digital computer. These problems were designed to test the theory developed, and to investigate the practical aspects of using the method. The results of the tests run are shown to coincide with the results predicted by the theory, and the estimates of error are shown to be conservative. Several techniques are developed which aid in the application of the method.

It is concluded that the locations of the poles and zeros of a rational function may be found by the method, with an accuracy limited only by the accuracy of the numerical calculations which must be made. It is recommended that the method be further developed along both theoretical and practical lines.

CHAPTER I

INTRODUCTION

The Problem.--An engineer may, at times, have on hand the experimentally measured frequency response of a network or physical system and desire to find an analytical expression to describe this response. The frequency response measured might be the gain of an amplifier, a driving point or transfer immittance of a network, or the transference of an automatic control system. A method of solving this problem was developed in this study.

Usual Solutions.--This problem is encountered frequently in automatic control work (1). It is generally necessary to devise a mathematical model of a component, such as a valve, or hydraulic actuator, before its effect on a system can be evaluated. Limited use can be made of experimental data directly; for example, Bode diagrams and Nyquist plots can be drawn. However, most of the more sophistocated methods of analysis and synthesis, such as the root-locus method and many of the statistical methods, require that the response of the system components be available in the form of an analytical expression. Even in cases where experimental data is directly usable, it is convenient to talk in terms of time constants, resonant frequencies and damping ratios, which are actually constants picked from the mathematical expression describing the response.

This problem is related to the approximation problem of network synthesis (2). The approximation problem is generally met in this form: Given a desired frequency response in graphical form, find an analytical expression which approximates the desired function as closely as necessary. Then realize this expression as a network. The crux of the problem lies in the selection of the analytical expression, and sound judgment is necessary here. First, it is necessary to select an expression which is, in fact, realizable. Second, the function must approximate the desired function within the tolerances set. Third, a function which satisfies the first two requirements will not necessarily be the simplest function which will do so, and it is possible that a less complicated network would result from selecting a different function.

The problem being considered here is a modification of the approximation problem. First, the correct form of the desired function is known. It is the ratio of two polynomials with real coefficients. Second, we know that an exact solution, not just an approximation, is possible. Third, the exact solution is unique, as will be shown, and there is no question of searching for a simpler solution.

Two general approaches to the problem are possible (3). One is to analyze the component under consideration on a theoretical basis to see what its transference ought to be. Then the actual transference of the component can be measured as a check on the calculations, and revisions can be made where necessary. The theoretical analysis can also be made only to discover the correct form for the mathematical expression. The experimental measurements are then used to evaluate the unknown constants. This theoretical approach would be used if sufficient information about the component were available to make the analysis possible. It is a practical method leading to good results, and the amount of time required is generally not prohibitive.

The second possible approach is to attempt to find the mathematical expression directly from the experimental data. This becomes necessary when it is not possible to make a theoretical analysis. This is a more difficult situation, since the engineer will not have the advantage of knowing what factors to expect in the analytical expression. This problem is usually solved graphically, by trial and error. In simple cases, the method is rapid and accurate. As the complexity of the component increases, however, the amount of time required increases, and the results of the process become more uncertain.

Characteristics of Response Functions.--It is now in order to review some of the characteristics of functions describing the response of physical systems (4, 5). The elements of which electrical and mechanical systems are composed are of three general types: those which store potential energy, those which store kinetic energy, and those which do not store energy, but dissipate it. (Energy stored in an inductance is usually considered to be "kinetic" energy, although there is no mathematical reason to do so.)

Let us consider now a general mechanical system. As a result of Newton's second law and Hooke's law, the velocity of any element is proportional to the force on the element or to the integral or derivative of the force. This is what is meant by a "linear" system. Thus, a set of simultaneous linear equations may be written involving the velocity of each element and the force on it. At this point an operational notation is usually introduced; when using the Laplace transform, $\frac{d}{dt}$ is replaced by s and $\int dt$ is replaced by $\frac{1}{s}$.* The set of equations may now be solved to obtain the velocity of any element as a function of the forces driving the system. The resulting output velocity is a linear combination of the output velocities due to each of the driving forces. The response due to each of the individual driving forces is in the form of the ratio of two polynomials in s multiplied by the transform of the driving force. The coefficients of s are products and sums of the system constants--

*Generally when the Laplace transform is introduced, information about initial conditions must be supplied. However, initial conditions affect only the transient solution. We are considering the steady state solution, and may therefore neglect initial conditions.

masses, spring constants, coefficients of friction--and so must be real. Such functions belong to the class termed rational functions (6).

To show the effect of a single driving force on the velocity at a certain point in a system, all other driving forces are set equal to zero, and the resulting velocity is divided by this driving force. The result is a "mechanical admittance." If the operator s is now considered to be the complex frequency variable $\sigma+j\omega$, then the mechanical admittance is a function of this complex variable--a class of functions which has been extensively studied. If the system is driven by a sinusoidal driving force of angular velocity ω, then the steady state response of the system may be evaluated by substituting $s=j\omega$ into the expression for the mechanical admittance. A similar discussion would show that response functions of electrical, thermal and other linear systems are also of the same form.

Immittance functions can be written in at least two additional forms, each of which emphasizes somewhat different aspects of the function. It should be pointed out that these are not different functions, but merely different forms of the same function. If the numerator and denominator polynomials are each factored and expressed as the product of a number of first degree factors (either numerator or denominator must also have a constant multiplier), then the function is said to be expressed in terms of poles and zeros. As a result of the restriction that the coefficients of the polynomials in the first form must be real, the poles and zeros of the immittance function must either occur on the real axis, or must occur in pairs symmetrical about the real axis.

A third form of the function results if the second form is expanded in partial fractions. Here the function is expressed as the sum of terms, each of which is a single pole function with a constant for a numerator.* These constants are the "residues" of the poles. This form of the function emphasizes the location of each pole, and the relative strength of each pole. The locations of the zeros are not apparent. Again, the poles must occur on the real axis, or in conjugate pairs. The residues of conjugate poles must themselves be conjugate, if the numerator polynomial in the first form is to have real coefficients.

Restrictions on the physical system will place additional restrictions on the immittance function. For instance, if the physical system is to be stable, then the immittance function must have no poles in the right half s plane, and only simple poles on the imaginary axis. If the physical system contains no internal sources of energy, then the argument of the function is restricted to $\pm\frac{\pi}{2}$. The method to be developed here, however, does not depend on such restrictions.

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*This is true if the poles are simple. The case of multiple poles is considered on p. 16.

Uniqueness of the Solution.--The problem might then be stated as follows: From a knowledge of a function of a complex variable along the positive imaginary axis, find the poles and zeros of the function. In this form, the problem is directly influenced by the principle of analytic continuation (7, 8). This principle states that if the value of a function and all of its derivatives are known at a single point, then the value of the function is uniquely determined throughout the region of analyticity of the function. It follows that if the value of a function is known along any finite line segment, then the function is uniquely determined, since the values of all of the derivatives may be determined for a point on the segment. Since in the problem at hand we know the function along the entire positive imaginary axis, it is uniquely determined through its region of analyticity--the entire s plane (excluding pole locations). Thus, the poles and zeros of the function are fixed, and the analytical expression is uniquely determined. No other solution is possible.

When experimental measurements are made of the response of a physical system, the information obtained is the magnitude and argument of the immittance function for values of s lying on the positive imaginary axis. The stated problem of this thesis, then, is to convert a knowledge of the value of a function along the positive imaginary axis to a mathematical expression which yields the same values along this axis. In view of the foregoing discussion, it is apparent that the problem is solved when the constants in any of the three forms of the function are evaluated. We may determine the polynomial coefficients, the locations of the poles and zeros (and the value of the constant multiplier), or the locations of the poles with the residue of each.

CHAPTER II

THEORETICAL ANALYSIS

General Solution.--The principle of analytic continuation guarantees that the solution to the problem, when found, will be unique, but is otherwise of little value in actually finding the solution. It is the power series which provides a starting point. A power series is an expression of the form

$$\sum_{n=0}^{\infty} c_n (s-s_o)^n,$$

and is an analytic function within its region of convergence (9). Every power series has a radius of convergence r such that the series converges absolutely when $|s-s_o|<r$, and diverges when $|s-s_o|>r$. The number r can be evaluated as follows:

$$r=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|, \text{ if the limit exists;} \qquad (1)$$

$$r=\lim_{n\to\infty}\frac{1}{\sqrt[n]{|c_n|}}, \text{ if the limit exists;} \qquad (2)$$

and in any case by the formula

$$r=\frac{1}{\overline{\lim_{n\to\infty}}\sqrt[n]{|c_n|}} . \qquad (3)$$

An arbitrary analytic function Z(s) may be represented in the vicinity of a point s_o by a power series, by letting

$$c_n = \frac{1}{n!} Z^{(n)}(s_o). \tag{4}$$

Then the function and the power series will be identical within the circle of convergence (10). The series will of course diverge outside of the circle of convergence. On the circle of convergence, $|s-s_o|=r$, the series may converge at all points, at some points, or at no points. However, there must be at least one singularity of the function Z(s) on the circle of convergence, and there must be no singularities within the circle of convergence (11).

It is thus apparent that if c_n from Equation 4 is substituted into Equation 1, 2, or 3, r will be the distance from the point s_o to the nearest singularity of Z(s). Performing this substitution, we obtain

$$r = \lim_{n\to\infty} \left| \frac{\frac{1}{(n-1)!} \; Z^{(n-1)}(s_o)}{\frac{1}{n!} \; Z^{(n)}(s_o)} \right|, \text{ if the limit exists;} \tag{5}$$

$$r = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\left|\frac{1}{n!} \; Z^{(n)}(s_o)\right|}} \quad, \text{ if the limit exists;} \tag{6}$$

$$r = \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{\left|\frac{1}{n!} \; Z^{(n)}(s_o)\right|}} \quad. \tag{7}$$

In the case we are considering, we are able to use Equations 5, 6 and 7 for any point on the positive imaginary axis (if the limit exists). If we now evaluate one of these expressions for a great number (infinite number) of points along the imaginary axis, and draw the circles of convergence about each point as in Figure 1, then the region inside of the envelope of these circles is known to be within the region of analyticity of the function Z(s). In general, it is uncertain at which points this envelope is in contact with a singularity of Z(s). When the envelope takes the form of an arc of a circle with center on the imaginary axis, a singularity may exist at any or all points on the arc. Singularities exist only at the ends of the arc a in Figure 1, but the singular region C borders a portion of the arc b. A singularity must exist at any point at which the envelope is not an arc of a circle with center on the imaginary axis. The singular region A causes the envelope to assume a shape other than the arc of a circle, as do the singularities at s_1 and s_2, and the curve B. If the function possesses isolated singularities near the imaginary axis, then they will cause cusps in the envelope. The singularities at s_1 and s_2 cause such cusps. A cusp, however, does not necessarily indicate that the singularity present is isolated, as is illustrated by the curve B, which also causes a cusp.

Unfortunately, the envelope will be symmetrical about the imaginary axis, since the centers of the circles of convergence all lie on the axis, and it will be uncertain whether a singularity lies in the left half plane, or at its mirror image in the right half plane. This ambiguity would have to be resolved by other means.

It is possible that other singularities of Z(s) farther from the imaginary axis will be "hidden." This is so, because in order for Equation 5, 6 or 7 to locate a singularity there must be a finite interval of the imaginary axis over which this is the closest singularity. For example, the singularity at s_3 is hidden by those at s_1 and s_2.

Application to Rational Functions.--It has been shown that Equations 5, 6 and 7 will yield the distance to the nearest singularity of a function, and that this information may be used to locate those isolated singularities of a function which lie nearest to the imaginary axis (except for a certain ambiguity). We shall now consider the case of rational functions. Since this is a more restricted function, with no singularities other than poles, we shall be able to develop a stronger result.

At this point we will choose one of the three available expressions for more detailed study. Because the expression selected must eventually be evaluated by numerical means, it seems wise to choose Equation 5, in order to avoid the necessity of obtaining a high order root of a complex number in rectangular form. It must be borne in mind that should the limit indicated fail to exist, then one of the other form must be selected.

The approach we shall use is to take Expression 5 as a hypothesis, and to evaluate it for a general rational function. We shall thus establish its validity independent of the foregoing discussion.

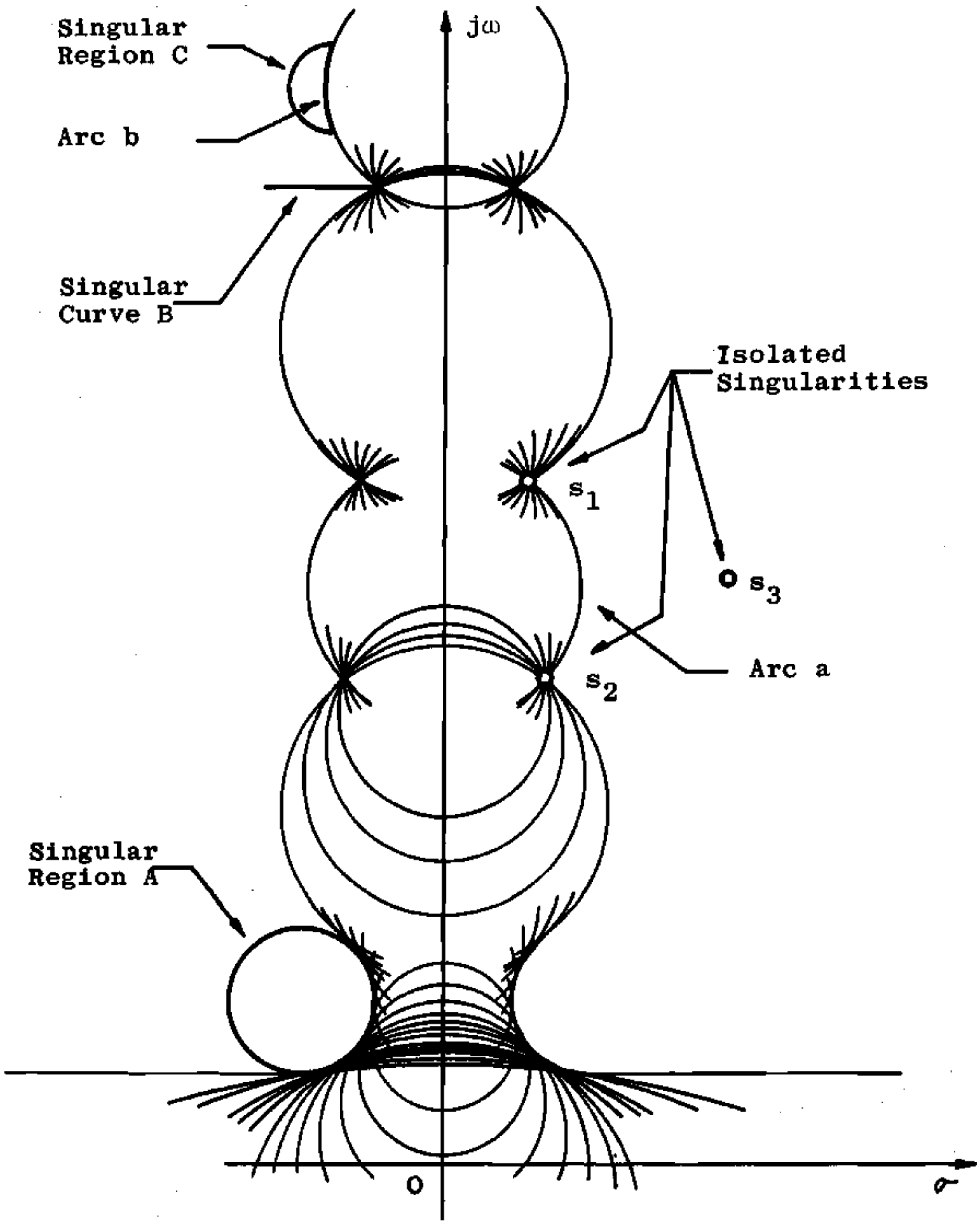

Fig. 1. Envelope of Circles of Convergence,

Illustrating Several Possible Effects of Singularities.

In order to use Equation 5 it is necessary to take successive derivatives of the function in question. This can best be done analytically if the function is in the form of the partial fraction expansion. For the present we shall consider only simple poles. We may represent a general rational function as (12)

$$Z(s) = \frac{k_1}{s-s_1} + \frac{k_2}{s-s_2} + \frac{k_3}{s-s_3} + \dots \tag{8}$$

Taking the n-th derivative of Z(s) and dividing by n!, we obtain

$$\frac{1}{n!} Z^{(n)}(s) = (-1)^n \left[\frac{k_1}{(s-s_1)^{n+1}} + \frac{k_2}{(s-s_2)^{n+1}} + \dots \right]. \tag{9}$$

Now notice that the terms within the brackets on the right hand side of Equation 9 contain $(s-s_1)^{n+1}$, $(s-s_2)^{n+1}$, etc. in their denominators. If n is large, any small difference in the magnitudes of the terms will be amplified. For instance,

$$\begin{aligned}
\text{if} \quad & \left|\frac{1}{s-s_1}\right| = 1.10 \left|\frac{1}{s-s_2}\right| , \\
\text{then} \quad & \left|\frac{1}{s-s_1}\right|^{3} = 1.33 \left|\frac{1}{s-s_2}\right|^{3} , \\
& \left|\frac{1}{s-s_1}\right|^{10} = 2.59 \left|\frac{1}{s-s_2}\right|^{10} , \\
& \left|\frac{1}{s-s_1}\right|^{20} = 6.72 \left|\frac{1}{s-s_2}\right|^{20} .
\end{aligned}$$

In fact, if one of the terms initially has a smaller denominator than the rest, n may be chosen large enough to make other terms negligible compared to this particular term.

Thus, the value of a high order derivative of a rational function depends primarily upon the nature and location of the nearest pole, and is almost unaffected by more distant poles. Provided that Expression 9 is not evaluated for a point equidistant from two poles, we may write

$$\frac{1}{n!} Z^{(n)}(s) = (-1)^n \frac{k_\nu}{(s-s_\nu)^{n+1}}(1+e), \qquad (10)$$

where $\lim_{n\to\infty} e = 0$.

Here s_ν is the location of the nearest pole, and k_ν is the residue in that pole. Substituting this expression into Equation 5, we obtain

$$r = \lim_{n\to\infty} \left| \frac{(-1)^{n-1} \dfrac{k_\nu}{(s-s_\nu)^n} \quad (1+e_1)}{(-1)^n \quad \dfrac{k_\nu}{(s-s_\nu)^{n+1}}(1+e_2)} \right| . \qquad (11)$$

Simplifying, this becomes

$$r = \lim_{n\to\infty} \left| (s_\nu - s)\frac{(1+e_1)}{(1+e_2)} \right| ,$$

$$r = |s_\nu - s| .$$

It has now been shown by a method independent of the previous more general argument that Equation 5 is valid for rational functions (for the case of simple poles). Furthermore, the limit indicated must exist, unless the point s_o is equidistant from two or more poles of Z(s).

Notice however, that the presence of the absolute magnitude sign in Equation 5 is unnecessary in the case of rational functions. Let us try Equation 5 without the absolute magnitude sign as a hypothesis, and test its validity, as we have just done for Equation 5:

$$\bar{r} = \lim_{n \to \infty} \left[\frac{\frac{1}{(n-1)!} \; Z^{(n-1)}(s_o)}{\frac{1}{n!} \; Z^{(n)}(s_o)} \right] . \tag{12}$$

Substituting Expression 10, we obtain

$$\bar{r} = \lim_{n \to \infty} \left[\frac{(-1)^{n-1} \dfrac{k_\nu}{(s-s_\nu)^n} \; (1+e_1)}{(-1)^n \; \dfrac{k_\nu}{(s-s_\nu)^{n+1}} (1+e_2)} \right] ,$$

$$\bar{r} = (s_\nu - s) .$$

It is thus clear that Equation 12 is valid for rational functions, and that the direction as well as the distance to the nearest pole is available. This resolves the ambiguity encountered earlier in attempting to locate poles, knowing only the distance to the pole.

Effect of Multiple Poles.--The case of functions containing multiple poles can now be considered. If a multiple pole is present, the partial fraction expansion of the function will include, in addition to terms for the simple poles, terms of the following form for each multiple pole (13):

$$Z(s)=\ldots+\frac{k_p}{(s-s_\nu)^p}+\frac{k_{p-1}}{(s-s_\nu)^{p-1}} \qquad (13)$$

$$+\ldots+\frac{k_2}{(s-s_\nu)^2}+\frac{k_1}{(s-s_\nu)}+\ldots ,$$

where p is the multiplicity of the pole. Taking the n-th derivative and dividing by n!, we obtain

$$\frac{1}{n!}Z^{(n)}(s)=\ldots+\frac{(-1)^n k_p (p)(p+1)(p+2)\ldots(p+n-1)}{n!(s-s_\nu)^{p+n}} \qquad (14)$$

$$+\frac{(-1)^n k_{p-1}(p-1)(p)(p+1)\ldots(p+n-2)}{n!(s-s_\nu)^{p+n-1}}$$

$$+\ldots+\frac{(-1)^n k_2 (2)(3)(4)\ldots(n+1)}{n!(s-s_\nu)^{n+2}}$$

$$+\frac{(-1)^n k_1 (1)(2)(3)\ldots(n)}{n!(s-s_\nu)^{n+1}} .$$

For $n \geqq p-1$, after cancelling common factors in numerator and denominator, we have

$$\frac{1}{n!}Z^{(n)}(s)=\ldots+\frac{(-1)^n}{(s-s_\nu)^{n+1}}\left[\frac{(n+1)(n+2)\ldots(n+p-1)k_p}{(p-1)!(s-s_\nu)^{p-1}}\right. \qquad (15)$$

$$+\frac{(n+1)(n+2)\ldots(n+p-2)k_{p-1}}{(p-2)!(s-s_\nu)^{p-2}}$$

$$\left.+\ldots+\frac{(n+1)(n+2)k_3}{2!(s-s_\nu)^2}+\frac{(n+1)k_2}{1!(s-s_\nu)}+k_1\right] .$$

Notice that the denominators inside the brackets are not functions of n. Notice that (if $|k_p| \approx |k_{p-1}|$) the numerator of the first term is (n+p-1) times as large as the second term, which is (n+p-2) times as large as the third, and so forth. If n is large, we may drop the second and subsequent terms:

$$\frac{1}{n!}\, Z^{(n)}(s) = \frac{(-1)^n k_p (n+1)(n+2)\ldots(n+p-1)}{(p-1)!\,(s-s_\nu)^{n+p}}(1+e), \qquad (16)$$

where $\lim\limits_{n\to\infty} e=0$.

For the case p=1, Equation 16 reduces to Equation 10, as it should. Now substitute Equation 16 into Equation 12.

$$\bar{r} = \lim_{n\to\infty}\left[\frac{\dfrac{(-1)^{n-1} k_p (n)(n+1)\ldots(n+p-2)}{(p-1)!\,(s-s_\nu)^{n+p-1}}\quad(1+e_1)}{\dfrac{(-1)^n \quad k_p (n+1)(n+2)\ldots(n+p-1)}{(p-1)!\,(s-s_\nu)^{n+p}}(1+e_2)}\right], \qquad (17)$$

$$\bar{r} = \lim_{n\to\infty}\left[(s_\nu - s)\frac{n}{(n+p-1)}\;\frac{(1+e_1)}{(1+e_2)}\right],$$

$$\bar{r} = (s_\nu - s).$$

We again find Equation 12 valid.

Though the multiple pole may be located by the same method used for simple poles, the presence of a multiple pole may adversely effect the operation of Equation 12 in locating other poles close to the multiple pole. This is because the numerator contains the factor $(n+1)(n+2)\ldots(n+p-1)$, which causes the numerator to increase with n faster than the denominator (for small n). Thus, it may be necessary to remove the multiple pole before the other poles in the vicinity will become apparent.

We now have a method of locating the poles of an impedance function from a knowledge of the value of the function along the imaginary axis. It is easy to see that this tool enables a complete solution to the problem being considered. By evaluating Expression 12 for numerous points along the imaginary axis, the poles nearest the axis may be located. Hidden poles may be found later by the same method after the poles originally found have been removed from the function. When the poles of the function have been located, the reciprocal of the function may be considered, and the same process repeated to locate the zeros of the function. After the poles and zeros of the function have been removed, the remainder will be a constant along the entire axis--the constant multiplier.

It is not necessary to use this method to locate poles or zeros on the imaginary axis (including the point at infinity). These poles and zeros may be found by inspection of the response function. It is well to remove these poles and zeros initially, so that they will not hide other poles and zeros nearby.

CHAPTER III

NUMERICAL METHOD AND ERRORS

Sources of Error.--Using the method of Equation 12 it is now theoretically possible to locate the poles and zeros of any impedance function from experimental measurements of its steady state response. However, in order to do this the operations of Equation 12 must actually be carried out--the derivatives must be taken, the division must be performed and the limit as $n \to \infty$ must be obtained. In addition, since we are relying on experimental data, we will not know the value of the impedance function at every point on the imaginary axis, but only at discrete points. Thus practical difficulties in applying Equation 12 become evident.

Errors enter the solution from three sources--impossibility of letting $n \to \infty$, inaccuracy of numerical methods for obtaining derivatives, and computational errors. Several decisions must be made before Equation 12 can be put to use. First, a decision must be made as to how high n will be carried. Second, a numerical method of taking derivatives of sufficient accuracy must be selected. It must be borne in mind that these operations must be carried out in complex numbers. Also, care must be exercised that accuracy is not lost in calculations.

Error Due to the Use of Finite Order Derivatives.--We will consider first an estimate as to how high n must be carried. An approach which might be used is to calculate the magnitudes of the various terms in the partial fraction expansion of $Z^{(n)}(s)$ as functions of n. Then the decision can be made as to how large the largest of the undesired terms can be relative to the desired term.

Removing the desired factor from the brackets in Equation 9, we obtain

$$\frac{1}{n!}\, Z^{(n)}(s) = \frac{(-1)^n k_\nu}{(s-s_\nu)^{n+1}} \left[\frac{k_1}{k_\nu}\left(\frac{s-s_\nu}{s-s_1}\right)^{n+1} + \ldots \right. \tag{18}$$

$$\left. + \frac{k_{\nu-1}}{k_\nu}\left(\frac{s-s_\nu}{s-s_{\nu-1}}\right)^{n+1} + 1 + \frac{k_{\nu+1}}{k_\nu}\left(\frac{s-s_\nu}{s-s_{\nu+1}}\right)^{n+1} + \ldots \right].$$

The error in calculating s_ν will, of course, be the least when the terms within the brackets are the smallest. The poles contributing the most error to the solution are those nearest to s_ν. If errors due to farther removed poles are neglected, and only errors due to the two nearest poles are calculated, Expression 18 becomes simpler.

Consider Figure 2. As the point s moves up the imaginary axis toward point s_a, the error due to the pole at $s_{\nu-1}$ decreases, since $\frac{s-s_\nu}{s-s_{\nu-1}}$ is decreasing. At the same time, the error due to the pole at $s_{\nu+1}$ increases, and at some point such as s_a, they become equal. It is not possible to state that this is the point at which the location of s_ν may be calculated with minimum error, since other considerations, such as the magnitude and angle of each residue, and the relative angles of the two error terms enter the picture. For purposes of calculating the error, we will assume that this is such a point. Now let us further simplify the picture, first by considering that the residues are all of roughly equal magnitude, and second by dropping the angles from the error terms. Then the expression for $Z^{(n)}(s)$ at point s_a becomes

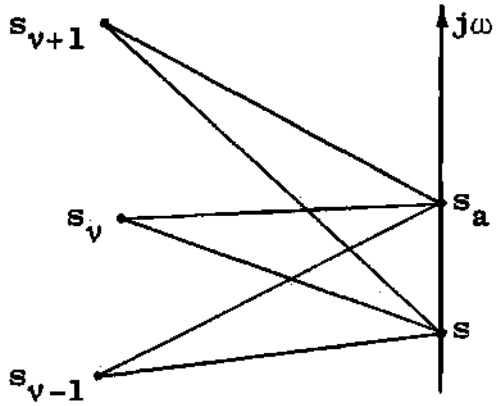

Fig. 2. Error Due to Adjacent Poles.

$$\frac{1}{n!} Z^{(n)}(s_a) = \frac{(-1)^n k_\nu}{(s_a - s_\nu)^{n+1}} \left[1 + 2 \left| \frac{s_a - s_\nu}{s_a - s_{\nu+1}} \right|^{n+1} \right] . \tag{19}$$

Similarly expressing $\frac{1}{(n-1)!} Z^{(n-1)}(s_a)$ and substituting into Equation 12 we obtain

$$s_\nu^* = s_a + \frac{1 + 2 \left| \frac{s_a - s_\nu}{s_a - s_{\nu+1}} \right|^n}{1 - 2 \left| \frac{s_a - s_\nu}{s_a - s_{\nu+1}} \right|^{n+1}} (s_\nu - s_a) , \tag{20}$$

where s_ν^* is the estimated value of s_ν. The sign of the error in the denominator has been made negative, so that the errors in the numerator and the denominator will not tend to cancel. This relation may be further simplified, using the relation $\frac{1}{1-e}\approx 1+e$, under the assumption that the error is small. Finally, assuming that

$$\left|\frac{s_a-s_\nu}{s_a-s_{\nu+1}}\right|^{2n+1} \ll \left|\frac{s_a-s_\nu}{s_a-s_{\nu+1}}\right|^{n},$$

we may write

$$\left|\frac{s_\nu{}^*-s_\nu}{s_\nu-s_a}\right| \leqq 4\left|\frac{s_a-s_\nu}{s_a-s_{\nu+1}}\right|^{n}. \tag{21}$$

This is no doubt a pessimistic view, since the assumption has been made that the four error terms, two in the numerator and two in the denominator will all add. To determine the location of a pole within 10 per cent of its distance from the imaginary axis, with $\frac{s_a-s_\nu}{s_a-s_{\nu+1}} = 0.8$, it is necessary that

$$4(0.8)^n \leqq 0.1;$$
$$(0.8)^n \leqq 0.025;$$
$$n=17.$$

Although we now have an expression for the accuracy of $s_\nu{}^*$, the problem of locating the point s_a remains. This will be considered later.

<u>Errors Due to Numerical Derivatives</u>.--Let us now consider means of taking the required derivatives. The accuracy with which the derivative of a tabulated function may be taken depends upon the spacing between the points, and upon the method used to take the derivative. In general, an interpolating function is chosen which passes through two or more of the tabulated points and the derivative of this function at any desired point is then considered to be the derivative of the tabulated function (14). In simple interpolation, such as is used in finding the logarithm of a number from a table, it is usual to consider that the function being interpolated is very nearly linear. That is, to approximate the function between two tabulated points by a straight line connecting these points. The same idea is involved in calculating a derivative as $\frac{\Delta y}{\Delta x}$, which is merely the slope of the straight line connecting the two points.

If linear interpolation is not sufficiently accurate, then a function may be found which produces a smooth curve through three or more of the tabulated points. This function is then considered to represent the tabulated function in the interval between tabulated points. Many different functions are useful as interpolating functions, such as trigonometric functions, polynomials and rational fractions. We choose to use polynomials because of their simplicity.

It seems that obtaining the derivative of a tabulated function would be tedious, indeed, if a new interpolating function must be fitted each time a derivative is to be taken. Actually, if the tabulated points are evenly spaced, (with respect to the independent variable), then most of the work can be done once and for all ahead of time. A table of the derivatives of interpolating polynomials is given by Milne (15). A portion of this table is given in the appendix with a short discussion of its derivation. Two formulas from this table are of particular interest to us.

Three-point central derivative:

$$y_1' = \frac{1}{2h}(-y_o + 0 + y_2) - \frac{h^2}{6} Y^{(3)} \tag{22}$$

Five-point central derivative:

$$y_2' = \frac{1}{12h}(y_o - 8y_1 + 0 + 8y_3 - y_4) + \frac{h^4}{30} Y^{(5)} \tag{23}$$

These formulas give the derivative of a tabulated function at the central point of three points, and at the central point of five points. The three-point formula is of course the familiar $\frac{\Delta y}{\Delta x}$. In the notation of this table, y_o, y_1, etc. are the ordinates at tabulated points spaced at intervals of h along the abcissa. The last term of each formula is the error term in shorthand form. For instance, $Y^{(3)}$ means the third derivative of y evaluated at some point b between y_o and y_2. Here y is the correct value of the function for which the interpolating polynomial is an approximation.

The correct location of the point b cannot be determined, so the only safe procedure is to choose b at a point which will make $Y^{(3)}$ as large as possible.

The question may well be asked: "Of what value is a formula which gives the error in calculating the first derivative of a function, in terms of the third derivative of the same function?" If nothing whatsoever is known about the third derivative of the function, Milne answers this question by saying that it is of no value at all. However, often, as in this case, at least an estimate may be made of the value of the third derivative. An estimate of the magnitude of the derivative of an impedance function taken at a point on the imaginary axis close to a pole may be made by making the somewhat audacious assumption that this pole is the only one influencing the derivative at this point. While this assumption is hard to justify for low values of n, it actually becomes nearly true for higher n, as was shown in Chapter II, page 15. According to Equation 10 the derivative becomes a maximum on the imaginary axis at $\omega = \mathrm{Im}\left[s_\nu\right]$. We will therefore choose this point as b. We may then evaluate the term $Y^{(n)}$ as

$$Y^{(n)} = \frac{\left|k_\nu\right| n!}{(\mathrm{Re}\left[s_\nu\right])^{n+1}} \quad . \tag{24}$$

The error in the three-point derivative is then equal to or less than

$$\frac{h^2}{6}\ \frac{3!\,\left|k_\nu\right|}{(\mathrm{Re}\left[s_\nu\right])^4}\,.$$

This knowledge is of little use here, since it does not enable us to calculate the error in the second and subsequent derivatives. To do so we would require a knowledge also of the change in error of the first derivative between adjacent points. However this formula does allow us to assess the advantage of proceeding to a more complicated means of obtaining derivatives, as is shown in the following example.

If the three-point formula gives insufficient accuracy with a point spacing (h) of 0.1, what improvement is gained by using more data-- that is, decreasing h to 0.05? Since the error term is $\frac{h^2}{6}\ Y^{(3)}$, dividing h by 2 divides the error by 4. What is the error if h=0.1 and a five-point formula for the central derivative is used? The error term is $\frac{h^4}{30}\ Y^{(5)}$, so the error is cut

$$\text{from}\qquad \frac{0.01}{6}\ \frac{6\left|k_\nu\right|}{(\mathrm{Re}\left[s_\nu\right])^4}\ ,$$

$$\text{to}\qquad \frac{0.0001}{30}\ \frac{120\left|k_\nu\right|}{(\mathrm{Re}\left[s_\nu\right])^6} = \frac{0.0004\left|k_\nu\right|}{(\mathrm{Re}\left[s_\nu\right])^6}\ .$$

--an improvement by a factor of 25, if $\mathrm{Re}\left[s_\nu\right] \approx 1$. If h is decreased to 0.05, error is cut by an additional factor of $2^4 = 16$. Error is now only $\frac{1}{400}$ of the original error.

This is an estimate of the maximum possible error in the first derivative. We can make no statement about the error in subsequent derivatives, except that it is nearly sure to be larger. We have, however, been able to show that the theoretical advantage of using a more accurate formula for derivatives is great. As will be seen in the next section, the potential accuracy of more complicated formulas may be lost due to computational errors.

<u>Error Due to Computation</u>.--It is evident that the amount of computation involved in solving a problem by this method is great, and that some form of automatic computing machinery must be used if the method is to be practical. The calculations performed in connection with this thesis were run on the IBM 650 Computer at the Rich Electronic Computer Center, Georgia Institute of Technology.

It is difficult to estimate the error which is caused by inaccurate computations for the same reason that it is difficult to estimate the error due to the use of a particular formula for obtaining the derivatives. That is, even if the magnitude of the error in the first derivative can be estimated, there is no indication of the error in subsequent derivatives. The use of floating point arithmetic will insure that the best possible accuracy is obtained from the computer, but the price for using this arithmetic is high, since floating point arithmetic requires about ten times as much machine time as fixed point arithmetic. (This is true of the IBM 650; other more sophistocated computers can perform calculations directly in floating point arithmetic, with little or no loss of speed.)

The basic difficulty with using fixed point arithmetic for this problem is that the taking of derivatives involves finding the difference between two nearly equal quantities. If the decimal point is positioned to prevent overflow of the input numbers, then the accuracy of the difference is poor. In fact, if the spacing of the input data points is decreased in an effort to increase the accuracy of the derivatives, then the accuracy of the computation is decreased, since the differences become smaller.

There seems to be no way to handle this problem analytically. In Chapter IV, some observations will be made concerning the accuracy obtainable with each type of arithmetic. Accurate solutions to problems have been obtained using both types.

One problem now remains in connection with the numerical application of Equation 12--that of deciding, "what is the best estimate of the location of the pole?" This can best be discussed after observing the operation of the method on several sample problems, since the exercise of some judgment is involved. It will therefore be considered in Chapter IV.

CHAPTER IV

SAMPLE PROBLEMS AND CONCLUSIONS

General Method Used.--The sample problems discussed in this chapter were designed to verify the theory of Chapters II and III and to investigate the practical aspects of using this method. A main computer program was prepared, using the type of arithmetic and method of obtaining derivatives which were to be tested. The pole and zero locations for a test function were selected, and an auxiliary program was written to calculate the real and imaginary parts of this function at evenly spaced points over a range of values of ω. This was used as the input to the main program, which would ordinarily be obtained for measurements. The main program was then run to see how accurately the pole and zero locations could be regained.

Two main programs were used. The first was written in the Bell General Purpose System (16), which uses floating point arithmetic with eight significant digits plus a two digit exponent. Complex numbers were handled in rectangular form. Derivatives were obtained by using a modification of the three-point formula. Here Δy and Δx were taken between adjacent data points, so that the derivatives were actually calculated for points midway between input data points. Thus, h in Equation 22 is half of the spacing between adjacent data points. In order to evaluate Equation 12 for n=1, then, it was necessary to interpolate between input data points to obtain the values of Z(s). Linear interpolation was used, and s_ν* was again calculated. The process was continued. This procedure is economical of input data, since only half as much data is needed as if the three-point formula were used directly. However, some inaccuracy is introduced due to the necessary interpolation.

The second main program used was written in machine language, using fixed point arithmetic. Calculations were made using two digits before and eight digits after the decimal point. The maximum value of the magnitude of the input data was normalized to approximately unity. Complex numbers were again handled in rectangular form. Derivatives were obtained from the three-point or five-point formula directly, with no interpolation necessary. Thus h in Equation 22 or 23 was equal to the spacing between input data points.

The functions tested were selected with a view toward answering several questions:

1. What method should be used to calculate derivatives?
2. What is the highest order derivative of sufficient accuracy?
3. Is fixed point arithmetic sufficiently accurate?
4. How may the best estimate of the location of a pole finally be made?

Tests of Theoretical Method.--As a preliminary check on the operation of Equation 12 a test was made of the single pole function

$$Z(s)=\frac{1}{s+1}$$

using floating point arithmetic and the three-point formula with h=0.05 (first main program). Input data was calculated from ω=-0.2 to ω=+2.2. s_ν* was calculated up through the sixth derivative. The accuracy of s_ν* decreased with increasing n, but was still within 2 per cent of the distance from the imaginary axis for the highest derivative used. Since no interfering poles were present, and since floating point arithmetic was used, the error present was due mainly to the method of obtaining derivatives.

In order to test the operation of Equation 12 on a function with a multiple pole, the function

$$Z(s)=\frac{1}{(s+1)^2}$$

was tested. Fixed point arithmetic was used, and the five-point formula with h=0.05 (second main program). Calculations were carried through the fifth derivative. The results of this run are partially plotted in Figure 3. Here s_ν* calculated for the same value of ω are connected by a solid line and s_ν* calculated for the same value of n are connected by a dashed line. The value of $\frac{n}{n+p-1}$ is indicated along with

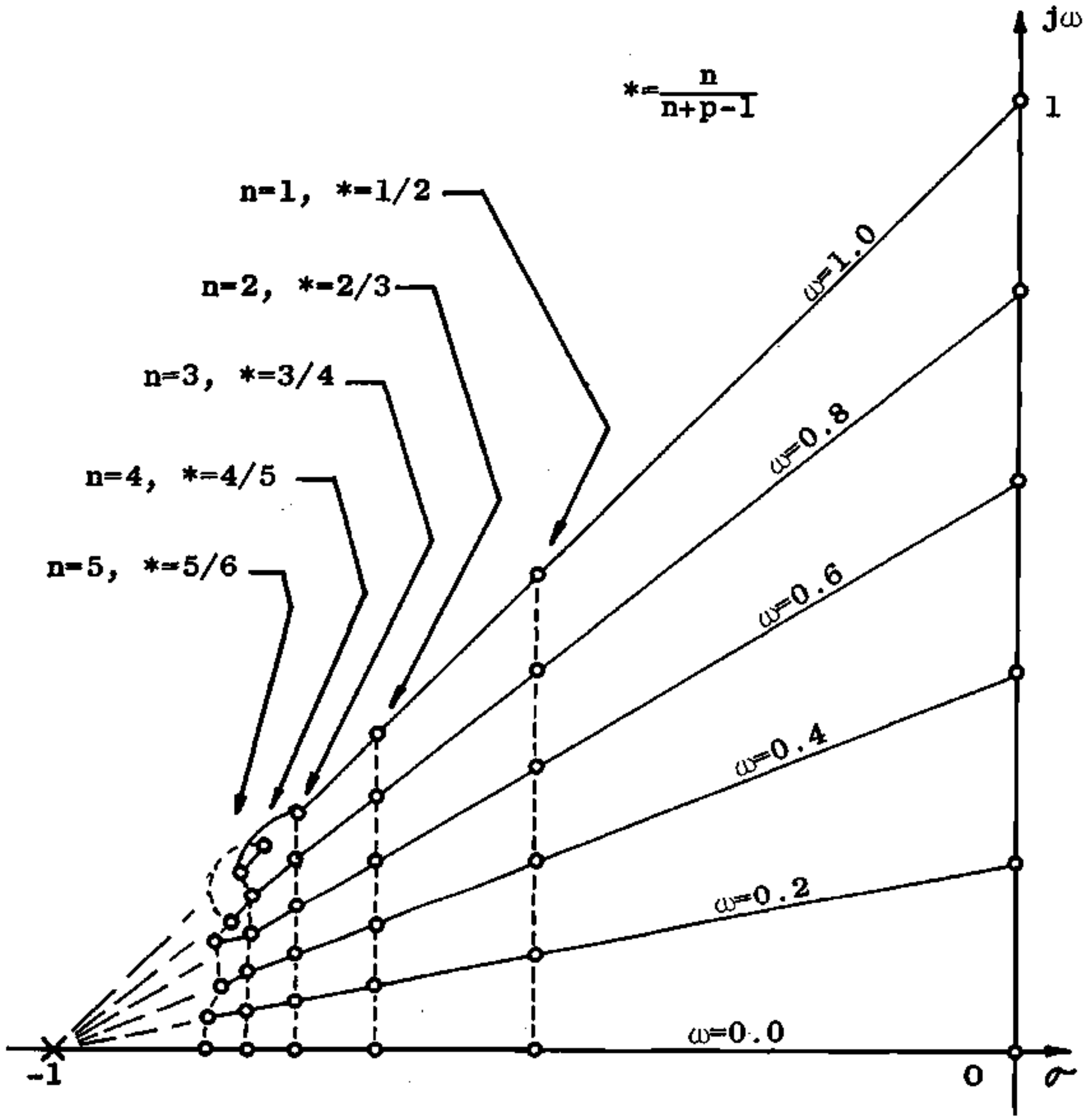

Fig. 3. Application to a Function with a Second-Order Pole.

the value of n. It can be seen that $s_v{*}-s=\frac{n}{n+p-1}(s_v-s)$, which verifies the conclusions of Chapter II concerning the convergence of Equation 12 in the presence of multiple poles. Errors due to computational inaccuracies begin to appear for n=5.

Equation 12 was tested on a three-pole function, so that the effect of interfering poles might be observed. The function selected was

$$Z(s)=\frac{1}{(s+1)(s+1+j)(s+1-j)}.$$

In order to prevent numerical errors from masking the effect of the interfering poles, a special program was written wherein derivatives were calculated analytically. If this function is expanded in the partial fraction form, and the n-th derivative is taken, we obtain

$$\frac{1}{n!}Z^{(n)}(s)=\frac{(-1)^n}{(s+1)^{n+1}}-\frac{1}{2}\,\frac{(-1)^n}{(s+1+j)^{n+1}}-\frac{1}{2}\,\frac{(-1)^n}{(s+1-j)^{n+1}}.$$

This form was used to calculate the values of successive derivatives of the function. Complex numbers were handled in polar form and floating point arithmetic was used. Calculations were made for values of ω from -0.4 to +2.4 at intervals equal to 0.1. Derivatives through the sixteenth were taken.

The results of this run are plotted in Figure 4. The salient feature of this figure is that successive values of $s_v{*}$ calculated for the same value of ω converge to the nearest pole as n increases. This is in agreement with the theory of Chapter II. It may also be noted that the convergence of s_ν* toward the pole is the more rapid as the ratio $\frac{s-s_\nu}{s-s_{\nu+1}}$ decreases. This is in agreement with the discussion concerning Figure 2. It is interesting to use this test run as a check on Equation 21. If we solve Equation 21 to find what value of n is necessary to locate the pole at -1+j0 within an error not to exceed 0.1, we obtain

$$\left|\frac{s_\nu{}^*-s_\nu}{s_\nu-s_a}\right| \leqq 4\left|\frac{1}{\sqrt{2}}\right|^n \left|\frac{0.1}{1}\right| \quad ;$$

$$(\sqrt{2})^n < 40 \quad ;$$

$$n=11 \quad .$$

Figure 4 shows that s_ν* was correct within 0.1 for n=7. Apparently Equation 21 is conservative, as was to be expected, considering the manner in which it was derived.

Investigation into Numerical Errors.--Since we have established that Equation 12 operates in accordance with the theory of Chapter II, we may now investigate the sources of error enumerated in Chapter III. The three-pole function just discussed was used as a test function. It was tested using floating and fixed point arithmetic, and the three- and five-point formulas for derivatives, with h equal to 0.05 and 0.1. The combinations are:

(1) fixed point, three-point formula with h=0.1,

(2) floating point, three-point formula with h=0.1,

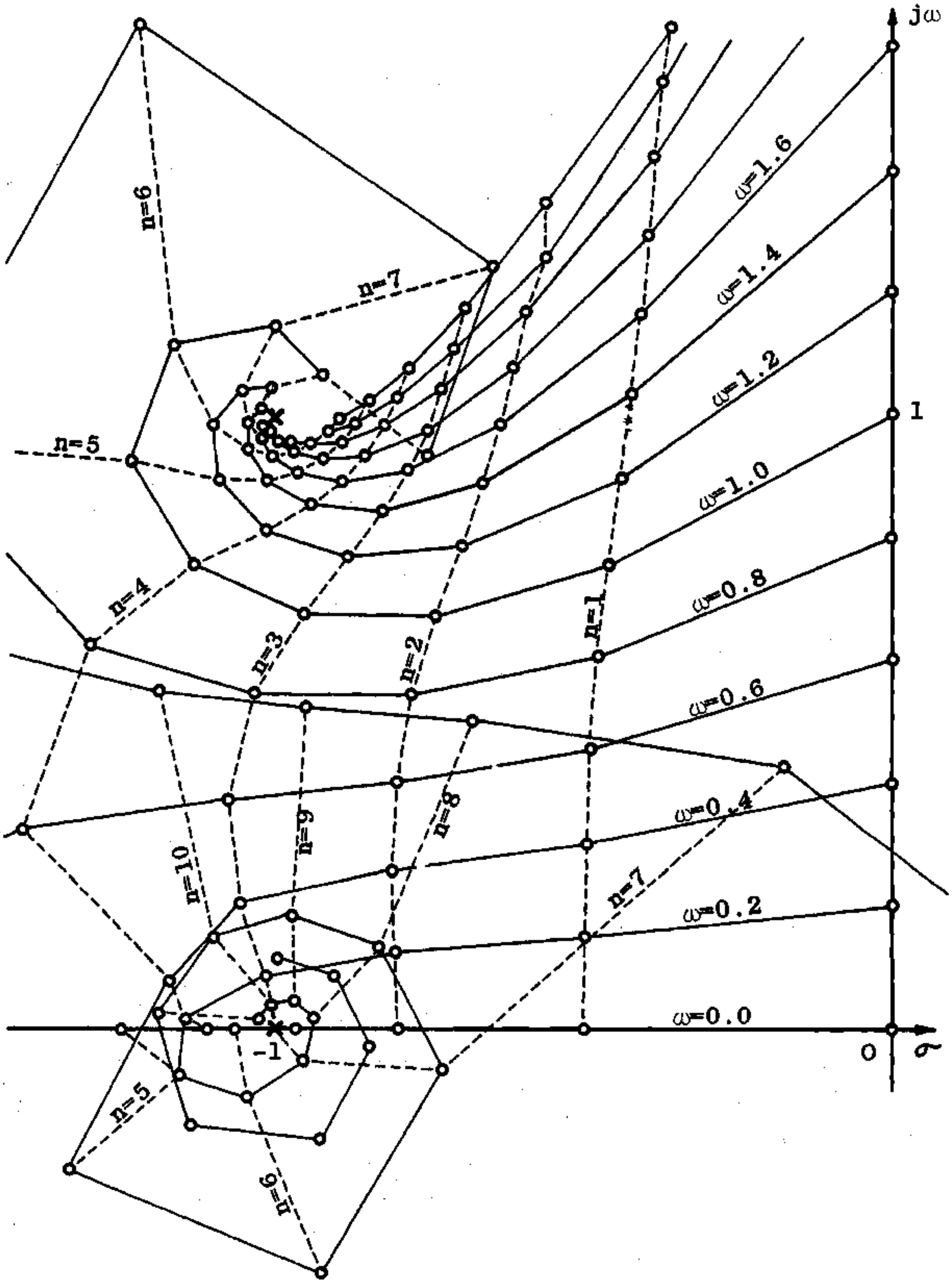

Fig. 4. Application to a Three-Pole Function; Derivatives Obtained Analytically.

(3) fixed point, three-point formula with h=0.05,

(4) floating point, three-point formula with h=0.05,

(5) fixed point, five-point formula with h=0.1.

For each of these runs except the last, computations were carried through the sixteenth derivative. The last was carried through the ninth derivative.

Figure 5 is a plot of the results of the first run listed above. It appears to be a distorted version of Figure 4. The effect of the errors present has been to shift the apparent locations of the poles, and to cause convergence to cease after about the seventh derivative. All of the runs made on this test function are plotted in Figures 6 and 7, for ω=0.2. The run using analytical derivatives (which is considered to be free from numerical error) is shown as a heavy line in both figures. Runs using fixed point arithmetic are shown in Figure 6, while runs using floating point arithmetic are shown in Figure 7.

Let us consider the effects of using more accurate derivatives, while retaining fixed point arithmetic. Considerable improvement is realized by decreasing h from 0.1 to 0.05. According to Equation 22 this should decrease the maximum possible error in the derivative by a factor of four. Figure 6 shows that the curve for h=0.05 does in fact appear to converge to a point closer to the pole than did the curve for h=0.1. Convergence seems to continue until about the ninth derivative. For higher derivatives, the curve becomes

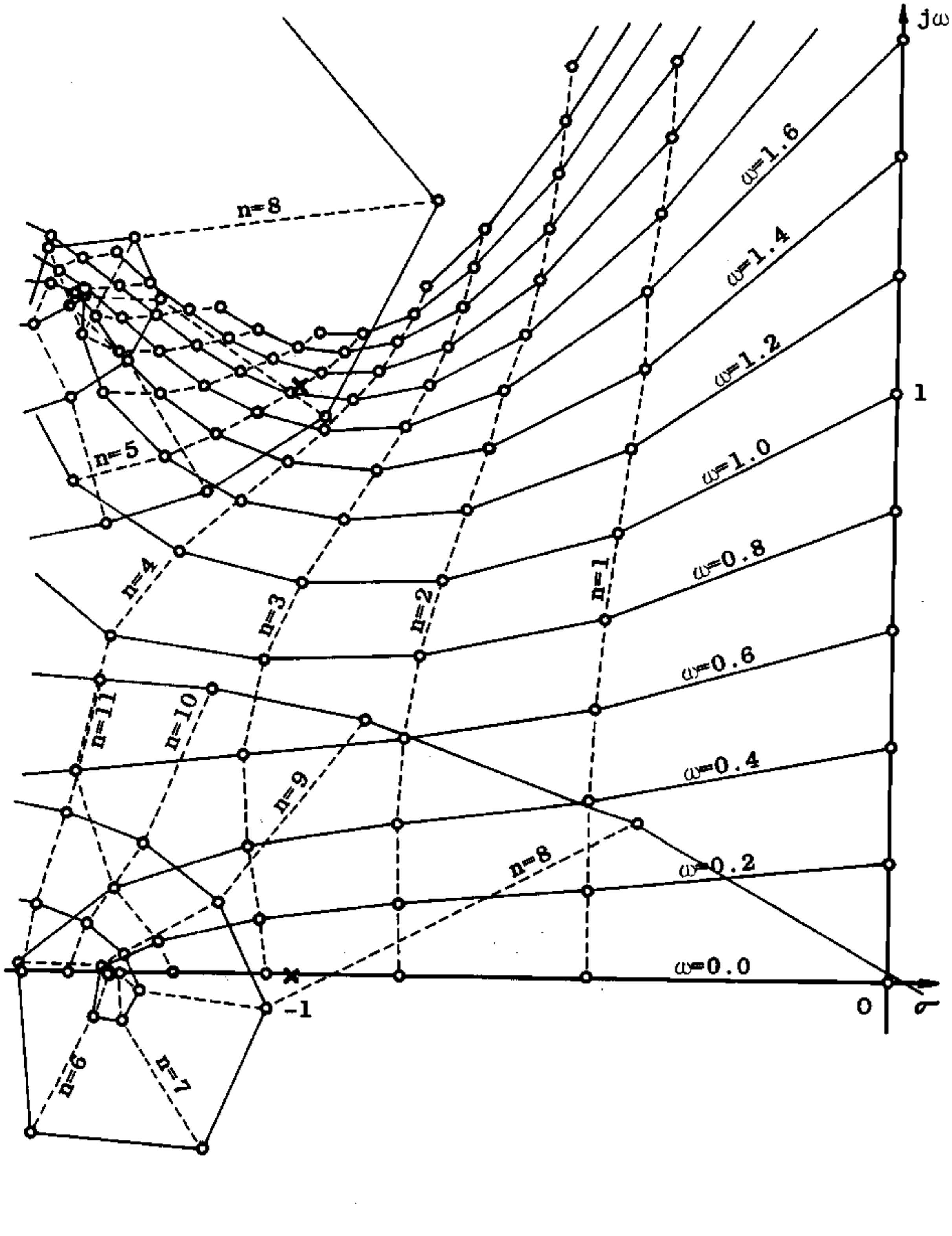

Fig. 5. Effect of Numerical Errors.

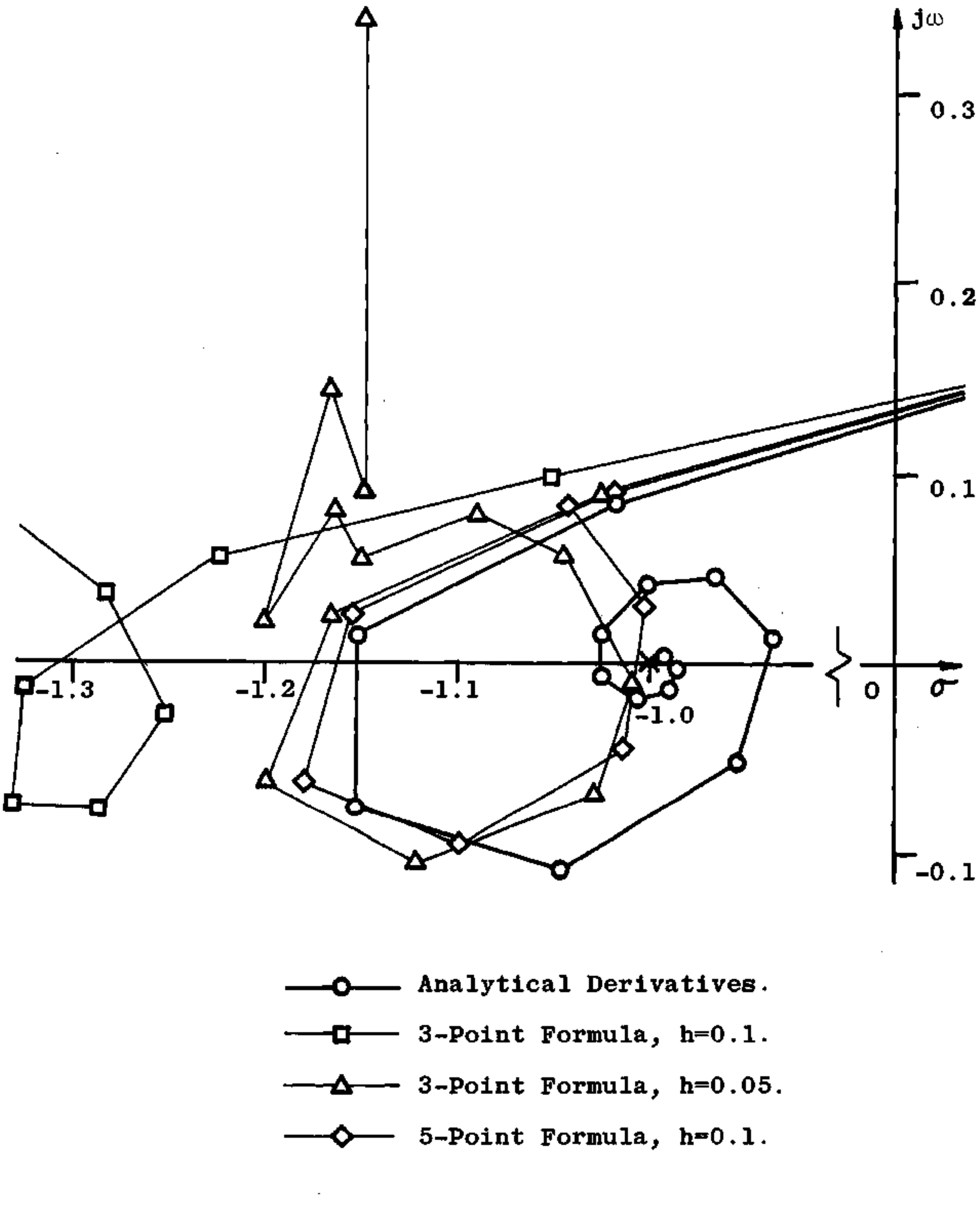

Fig. 6. Application to a Three-Pole Function; Derivatives Obtained by Fixed Point Arithmetic.

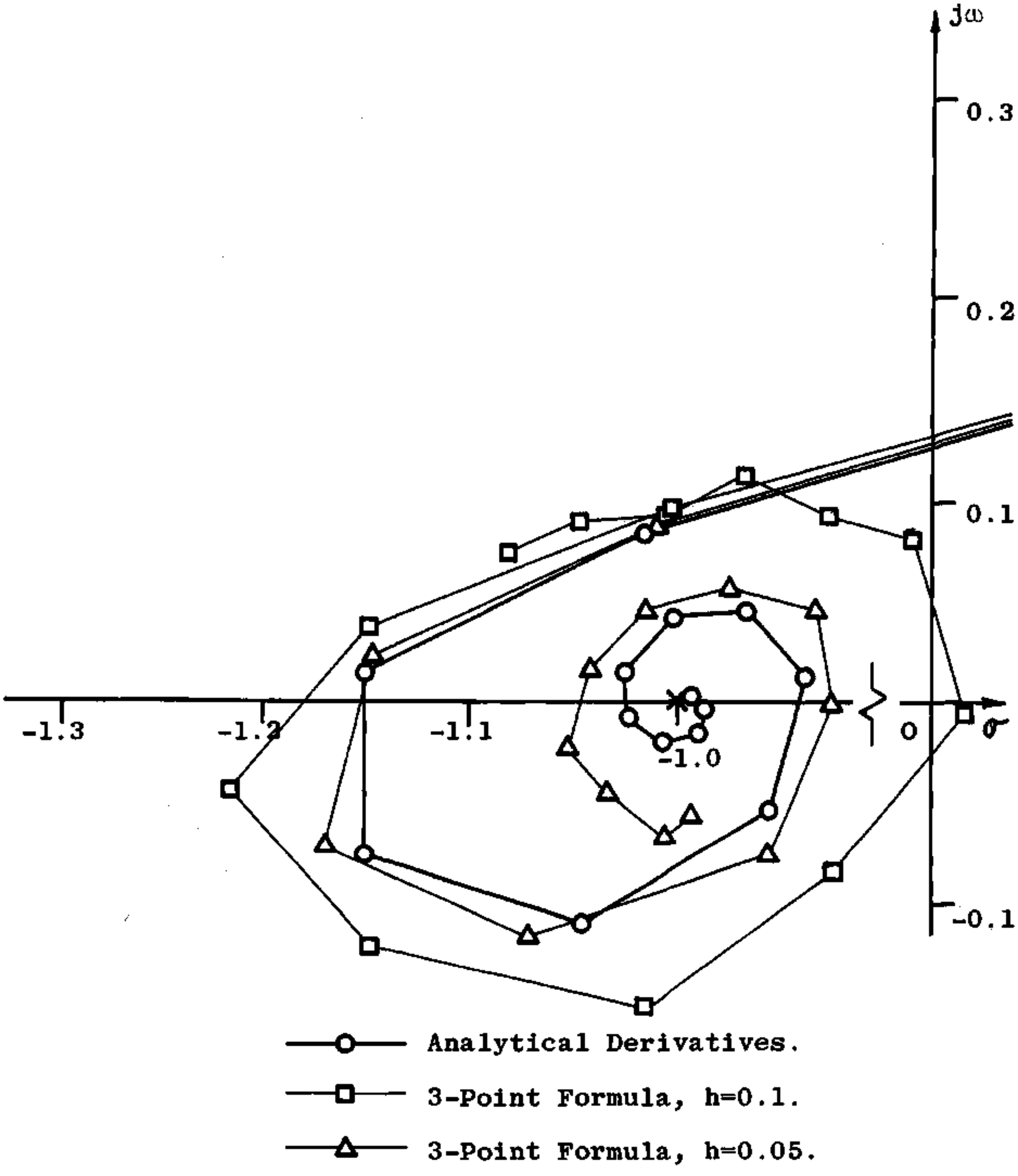

Fig. 7. Application to a Three-Pole Function; Derivatives Obtained by Floating Point Arithmetic.

completely erratic and meaningless. If the five-point derivative is used with h=0.1, in an effort to improve the accuracy of calculations, Figure 6 shows that no improvement results, though an improvement by a factor of about 25 was to be expected (p. 27). We may therefore conclude that the error in the curve for the three-point derivative with h=0.05 was largely due to the use of fixed point arithmetic.

Consider now the improved accuracy to be gained by the use of floating point arithmetic. Figure 7 shows that if floating point arithmetic is used, the apparent location of the pole is approximately correct and convergence is considerably better than was obtained using fixed point arithmetic. Convergence ceases after the ninth or tenth derivative for h=0.1 and after about the twelfth derivative for h=0.05. The run using floating point arithmetic and the three-point formula with h=0.05 was the most accurate one made on this function. No runs were made with the five-point formula using floating point arithmetic.

<u>Estimating Pole Locations</u>.--Up to this point we have considered a method of locating poles of an impedance function, and the limitations on the method which prevent us from finding the exact pole locations. However, in spite of these inaccuracies, one must finally select one point as the most probable position of each pole. This selection must be based primarily on judgment. The selection of the point s_a, which was mentioned in the determination of errors in Chapter III

(p. 21), is involved in estimating the pole locations.

A plot such as Figure 4 or Figure 5 presents all of the information which is available, and is very useful. From such a plot areas of convergence, each of which is likely to contain a pole, may be located. Then the values of ω for which convergence is most rapid may be selected. These values are the points s_a. The plot can then be examined to determine the order of the derivative n for which convergence ceases. It is then best to choose s_v* with $s_a=j\omega$, for the value of n selected above, as the estimated location of the pole. Alternatively, the apparent center of the area of convergence could be chosen as the estimated pole location. In the case illustrated in Figure 7, this would result in a more accurate choice. Practically, the exact location chosen is not extremely important unless the poles are closely spaced, since it is possible to refine the original estimates, as will be seen in the next section.

Considerable effort is required to make a plot such as Figure 4 or 5, and it would be desirable to avoid this labor, if possible. One possible means of avoiding the use of such a plot is based on the fact that for a given value of n, values of s_v* calculated for successive values of ω become more closely spaced near a pole. Accordingly, the second main program was modified to compute the spacing between successive points, $|\Delta s_v*|$. The tabulated data from this program can be scanned, and values of ω for which $|\Delta s_\nu *|$ is a minimum can be selected. As the data is scanned for higher derivatives, $|\Delta s_\nu *|$ decreases, and the value of ω where it is a minimum becomes more distinct. Finally, as yet higher derivatives are scanned, $|\Delta s_\nu *|$ no longer decreases and may become erratic. This means that $s_\nu *$ is no longer converging with increasing n. The estimated pole location can then be chosen as the value of $s_\nu *$ for which $|\Delta s_\nu *|$ is a minimum.

It should be noted that this abbreviated method of estimating the pole locations is based on the empirical observation that $|\Delta s_\nu *|$ decreases near a pole. Although it has been borne out in all cases in which it has been investigated, it might not always be reliable.

Refining Estimated Locations.--Having made a preliminary estimate of the locations of the poles, we may now refine this estimate as far as is desired. This is done by placing a zero at all of the estimated locations except one, and then re-running this as a new problem. The presence of a zero near each pole except one will reduce the interference due to these poles and will allow the remaining pole location to be more accurately determined. This is possible, because the presence of a zero near a pole reduces the residue in that pole: For example

$$\frac{1}{(s+1)(s-1)} = \frac{-0.5}{s+1} + \frac{0.5}{s-1} ;$$

$$\frac{(s+1.1)}{(s+1)(s-1)} = \frac{-0.05}{s+1} + \frac{1.05}{s-1} .$$

In this manner the accuracy of each estimated pole location can be improved. Zeros can then be placed at the new estimated locations, and the process can be repeated until the desired accuracy is obtained.

This process was tested on the three-pole function previously mentioned. A preliminary run using fixed point arithmetic and the five-point formula with h=0.05 showed poles at approximately

$$s_1 = -0.9622-j0.8447,$$

$$s_2 = -1.1122+j0,$$

$$s_3 = -0.9622+j0.8447.$$

These values are in error by about 15 per cent of their distance from the imaginary axis. The input data was multiplied by $(s-s_1)(s-s_2)$, and the resulting function was re-run up through the fourth derivative. (The modified function does not have zeros symmetrically located with respect to the real axis, and thus could not be an impedance function, but this does not affect the operation of the method.)

The function behaved very much like a single pole function, and the pole at s_3 was re-estimated at -1.00867 +j0.99642. The error in location is now only about 1.0 per cent of the distance from the imaginary axis. The next step would have been to refine the estimated location of the real pole. However, this problem was carried no further.

This method of refining estimated pole locations also applies to multiple poles, but with reservations. As a test, a function with a double pole at -1+j0 was chosen. A zero was placed at -0.9+j0 (as if this had been the first estimate of the pole location) and the function was run. The function behaved like a single pole function, but with the pole at about -1.11+j0. This is not surprising, though, since it is well known that several closely spaced poles may be replaced by one multiple pole at their center of gravity, with little change in the value of the function on the imaginary axis. It is theoretically important that this method provides no way to distinguish a multiple pole from a group of closely spaced simple poles, though it is of little practical consequence.

Final Test Problem.--The three-pole function was finally tested with three zeros added at -2+j1, -2+j0, and -2-j1. This was done to insure that the presence of the zeros would not adversely affect the method. Fixed point arithmetic was used and the five-point formula with h equal to 0.05. The results of the test on the function are summarized as follows:

First estimate of pole locations:

$$s_1 = -0.9571+j1.0644,$$

$$s_2 = -0.8718+j0,$$

$$s_3 = -0.9571-j1.0644.$$

Second estimate of pole locations:

$$s_1 = -0.999918+j1.005859,$$

$$s_2 = -1.039529-j0.031116,$$

$$s_3 = -0.999918-j1.005859.$$

Third estimate of real pole location:

$$s_2 = -1.002950+j0.000289.$$

The function was inverted and re-run.

First estimate of zero locations:

$$s_4 = -1.7233+j1.4282,$$

s_5 was indistinct,

$$s_6 = -1.7233-j1.4282.$$

First estimate of real zero location:

$$s_5 = -1.7116+j0.0027.$$

Second estimate of complex zero location:

$$s_4 = -1.9100+j1.1588,$$

$$s_6 = -1.9100-j1.1588.$$

The problem was carried no further, since the zero locations were converging to the correct values.

Conclusions.--We are now in a position to answer, in a general way, the questions posed at the beginning of this chapter.

1. What method should be used to calculate derivatives? The accuracy of the three-point formula was adequate

for the problems that have been run. There appears to be no reason to go to the five-point formula if fixed point arithmetic is used, since its potential accuracy is lost in computational errors. Indeed, it is doubtful whether the use of the five-point formula would be justified at all, since the accuracy of the solution would ultimately be limited by the accuracy of the experimental data.

2. What is the highest order derivative of sufficient accuracy to be usable? The point at which convergence ceases has been apparent from plots such as Figures 6 and 7 in all cases investigated. When this point is reached, there is no reason to take further derivatives.

3. Is fixed point arithmetic sufficiently accurate? For the sample problems which have been run, fixed point arithmetic was sufficiently accurate. Again, it is doubtful whether the use of floating point arithmetic would be advantageous, because of the limited accuracy of the experimental input data.

4. How may the best estimate of the location of a pole finally be selected? This has been discussed (p. 41), and two methods based on observations of sample problems have been suggested. In connection with this problem, it should be pointed out that the initial run on a problem is crucial. If the estimated pole locations obtained from this run are sufficiently accurate, then they may be refined as far as is desired by subsequent runs. Therefore, it is desirable to

take enough derivatives on the first run to insure that the point where convergence ceases will have been reached. It would probably be desirable to make a plot such as Figure 4 before choosing the estimated pole locations. For subsequent runs, with only one strong pole present, only three or four derivatives need be taken.

Let us now summarize what has been accomplished on this problem. In Chapter II a theoretical analysis was made which indicated that the locations of the poles of a function could be obtained by the application of Equation 12. It was shown that this equation holds for simple or multiple poles. In Chapter III the practical aspects of the applications of Equation 12 were investigated, and some estimates of the errors involved were made. In Chapter IV, sample problems were discussed which verified the results of the previous chapters, and certain practical observations about the application of the method were made.

The treatment of the subject presented here has by no means been exhaustive, and further investigation would certainly be profitable. Following a theoretical approach it would be interesting to pursue the application of Equations 5, 6 and 7 to functions not restricted to isolated singularities. This might lead to a solution to the approximation problem, wherein a given response function is realized by an infinite number of poles spaced along a line which would be located by the method. The function would then be approximated by a finite number of poles spaced along this line. Practically, it would be useful to find better estimates of the errors involved in the application of the method, and to investigate the resolving power of Equation 12 using specific means of obtaining derivatives. The effect of the accuracy of the input data should also be studied, and the accuracy possible with a given number of input data points should be evaluated. It would also be interesting to apply Equation 6 to some sample problems, since it might well lead to more accurate results. It would also be useful to investigate the practicality of obtaining the required derivatives by the use of a different interpolating function, such as a rational function, or by analog means.

APPENDIX

FORMULAS FOR THE DERIVATIVE OF A TABULATED FUNCTION

The following is taken from Milne (15).

A tabulated function is known accurately at discrete (usually evenly spaced) points, only. If it is necessary to find the value of the function at intermediate points, then a "smooth" curve is faired through two or more of the tabulated points. This curve represents an estimate of the value of the function between tabulated points.

Many functions are useful as interpolating functions. Trigonometric functions may be used for interpolating periodic functions. Rational fractions are useful for interpolating in the vicinity of a pole of the tabulated function. However, polynomials are most often used for interpolation, because of their simplicity.

A polynomial of degree n will have n+1 constants, and by proper choice of the constants, may be made to pass through n+1 points of a single valued tabulated function. If x_o, y_o; x_1, y_1; x_2, y_2 are three points of a tabulated function, then it is apparent by inspection that the following function will pass through these points:

$$y = \frac{y_o(x-x_1)(x-x_2)}{(x_o-x_1)(x_o-x_2)} + \frac{y_1(x-x_o)(x-x_2)}{(x_1-x_o)(x_1-x_2)}$$
$$+ \frac{y_2(x-x_o)(x-x_1)}{(x_2-x_o)(x_2-x_1)} .$$

This function is a second-degree polynomial. The form may easily be extended to any desired number of points.

If the derivative of the tabulated function is required, it may be found by differentiating the interpolating function. In the case of the second degree polynomial this becomes:

$$y' = \frac{y_o\left[(x-x_1)+(x-x_2)\right]}{(x_o-x_1)(x_o-x_2)} + \frac{y_1\left[(x-x_o)+(x-x_2)\right]}{(x_1-x_o)(x_1-x_2)}$$
$$+ \frac{y_2\left[(x-x_o)+(x-x_1)\right]}{(x_2-x_o)(x_2-x_1)} .$$

This is a complicated function, and is inconvenient to evaluate. However, if the tabulated points are evenly spaced along the abcissa (the usual case) this simplifies to:

$$y' = \frac{y_o\left[2(x-x_o)-3h\right]}{2h^2} + \frac{y_1\left[2(x-x_o)-2h\right]}{-h^2}$$
$$+ \frac{y_2\left[2(x-x_o)-h\right]}{2h^2} ,$$

where h is the spacing between points. If the derivative is required only at the tabulated points, this further simplifies to:

$$y_o' = \frac{1}{2h}(-3y_o + 4y_1 - y_2)$$

$$y_1' = \frac{1}{2h}(- y_o + 0 + y_2)$$

$$y_2' = \frac{1}{2h}(y_o - 4y_1 + 3y_2).$$

These formulas are quite easy to use. The formula for the derivative at the central point of an odd number of points is called by Milne a "central derivative" formula. The value of the function at the central point does not affect the value of the derivative at that point. The central derivative therefore requires one less constant to be evaluated than derivatives at other points. The central derivative is also more accurate. In the case illustrated above, the central derivative is merely $\frac{\Delta y}{\Delta x}$.

It is difficult to estimate the accuracy of an interpolated value of a function unless the correct value of the function is also known. Then, of course, the interpolation is unnecessary. In general it may be said, however, that the smaller the higher derivatives of the function are, the less violently the function will behave between tabulated points, and the more accurate a smooth curve approximation will be. In fact, if all derivatives of the function above the n-th vanish, then an n-th degree polynomial will describe the function exactly.

Milne has developed an estimate of the maximum error in the derivative of a tabulated function. The use of his error formula requires a knowledge of the maximum value of the (n+1)-st derivative of the tabulated function. This information is sometimes available, as for example, in the case of a sine function. The error formula will not be derived here, but the error term is included in the table below.

The following is a portion of a table of formulas giving the derivative of a tabulated function at one of the tabulated points. The table referenced covers formulas using from three through eight tabulated points.

$n=2$:

$$y_o' = \frac{1}{2h}(-3y_o + 4y_1 - y_2) + \frac{h^2}{3}Y^{(3)}$$

$$y_1' = \frac{1}{2h}(-y_o + 0 + y_2) - \frac{h^2}{6}Y^{(3)}$$

$$y_2' = \frac{1}{2h}(y_o - 4y_1 + 3y_2) + \frac{h^2}{3}Y^{(3)}$$

$n=4$:

$$y_o' = \frac{1}{12h}(-25y_o + 48y_1 - 36y_2 + 16y_3 - 3y_4) + \frac{h^4}{5}Y^{(5)}$$

$$y_1' = \frac{1}{12h}(-3y_o - 10y_1 + 18y_2 - 6y_3 + y_4) - \frac{h^4}{20}Y^{(5)}$$

$$y_2' = \frac{1}{12h}(y_o - 8y_1 + 0 + 8y_3 - y_4) + \frac{h^4}{30}Y^{(5)}$$

$$y_3' = \frac{1}{12h}(-y_o + 6y_1 - 18y_2 + 10y_3 + 3y_4) - \frac{h^4}{20}Y^{(5)}$$

$$y_4' = \frac{1}{12h}(3y_o - 16y_1 + 36y_2 - 48y_3 + 25y_4) + \frac{h^4}{5}Y^{(5)}$$

where $h = \Delta x$,

$$Y^{(n)} = \frac{d^n y}{dx^n} \text{ at a point } b;\ x_o \leq b \leq x_{max}.$$

Point b is chosen to make $Y^{(n)}$ as large as possible.

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